

A functional limit theorem for irregular SDEs

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Let X_1, X_2, \dots be a sequence of i.i.d. real-valued random variables with mean zero, and consider the scaled random walk of the form $Y_{k+1}^N = Y_k^N + a_N(Y_k^N)X_{k+1}$, where $a_N : \mathbb{R} \rightarrow \mathbb{R}_+$. We show, under mild assumptions on the law of X_i , that one can choose the scale factor a_N in such a way that the process $(Y_{\lfloor Nt \rfloor}^N)_{t \in \mathbb{R}_+}$ converges in distribution to a given diffusion $(M_t)_{t \in \mathbb{R}_+}$ solving a stochastic differential equation with possibly irregular coefficients, as $N \rightarrow \infty$. To this end we embed the scaled random walks into the diffusion M with a sequence of stopping times with expected time step $1/N$.

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Introduction

Let X_1, X_2, \dots be a sequence of i.i.d. integrable random variables with $E(X_i) = 0$. Let $a_N : \mathbb{R} \rightarrow \mathbb{R}_+$ be a function depending on $N \in \mathbb{N}$, and let $(Y_k^N)_{k \in \mathbb{Z}_+}$ be the process

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satisfying $Y_0^N = m \in \mathbb{R}$ and

$$Y_{k+1}^N = Y_k^N + a_N(Y_k^N)X_{k+1}, \quad k \in \mathbb{Z}_+. \quad (1)$$

We extend Y^N to a continuous time processes by defining $Y_t^N = Y_{[t]}^N + (t - [t])(Y_{[t]+1}^N - Y_{[t]}^N)$.

Consider the particular case where $E(X_i^2) = 1$ and a_N is constant equal to $\frac{1}{\sqrt{N}}$. Then (Y_k^N) is the random walk generated by (X_i) , scaled by the constant $\frac{1}{\sqrt{N}}$, and Donsker's theorem implies that the continuous-time process $(Y_{Nt}^N)_{t \in \mathbb{R}_+}$ converges in distribution to a Brownian motion as $N \rightarrow \infty$ (see e.g. [3], [10] or Section 8.6 in [4]).

In this paper we address the question of whether we can choose the scale factor a_N in such a way that the scaled random walk $(Y_{Nt}^N)_{t \in \mathbb{R}_+}$ converges in distribution to a time homogeneous diffusion M satisfying the stochastic differential equation (SDE)

$$dM_t = \eta(M_t)dW_t, \quad M_0 = m, \quad (2)$$

where W is a Brownian motion, and $\eta: \mathbb{R} \rightarrow \mathbb{R}$ is a Borel-measurable function that satisfies the Engelbert-Schmidt conditions (see [5]) in some interval (l, r) , $-\infty \leq l < r \leq \infty$, and vanishes outside (l, r) .

If convergence takes place, then one can use the limiting process M as a proxy for the scaled random walk Y^N for large N ; or vice versa, Y^N can be used for approximating the SDE M . One can thus profit from tools for continuous-time *and* discrete-time processes for analyzing both processes M and Y^N .

If η is Lipschitz continuous, then a natural choice for the scale factor is $a_N(y) = \frac{1}{\sqrt{N}}\eta(y)$. Then (Y_k^N) can be interpreted as the Euler approximation of M , and it is known that it converges in distribution to M (see e.g. [8]). For *arbitrary* diffusion coefficients η satisfying the Engelbert-Schmidt conditions the question of whether there exist scale factors such Y^N converges to M has not been solved. If the diffusion coefficient is very irregular, then the diffusion intensity $\eta(x_0)$ at a fixed state point x_0 can not be used as an approximation of the diffusion coefficient in the neighborhood of x_0 . Therefore, in order to have convergence, the scaling factors a_N need to take into account the global structure of η .

Recall that Skorokhod proves Donsker's theorem by embedding in law the random walk scaled by the constant $\frac{1}{\sqrt{N}}$ into the Brownian motion with a sequence of stopping times (see [11]). We take on Skorokhod's idea and show, under some nice conditions on the distribution of X_i , that there exists a scale factor $a_N: (l, r) \rightarrow (0, \infty)$ such that $(Y_k^N)_{k \geq 0}$ can be embedded into the diffusion M with a sequence of stopping times with expected time step $1/N$.

Loosely speaking, the embedding works as follows. We first choose $a_N(m)$ (recall that m is the starting point in (2)) and a stopping time ρ_1 such that $E(\rho_1) = 1/N$ and $M_{\rho_1} \stackrel{d}{=} Y_1^N$. Conditionally on $\{M_{\rho_1} = y\}$ we choose $a_N(y)$ and a stopping time ρ_2 such that $E(\rho_2) = 1/N$ and $M_{\rho_1+\rho_2} \stackrel{d}{=} y + a_N(y)X_2$. By proceeding like this we obtain a sequence of stopping times $\tau_k = \rho_1 + \dots + \rho_k$ such that $(M_{\tau_k})_{k \geq 0}$ has the same distribution as the scaled random walk $(Y_k^N)_{k \geq 0}$.

The times ρ_k turn out to be pairwise uncorrelated and we can check that they satisfy a certain uniform integrability property (see Lemma 3.3). Under such a uniform integrability property we prove a version of the weak law of large numbers for uncorrelated arrays, which is also interesting in itself because we do not require finiteness of the second moments (see Theorem 3.4). This weak law of large numbers entails that for all $t \in \mathbb{R}_+$ we have $\tau_{\lfloor Nt \rfloor} \rightarrow t$ in probability, as $N \rightarrow \infty$. From this, one can deduce that $(M_{\tau_{\lfloor Nt \rfloor}})_{t \in \mathbb{R}_+}$ converges in probability to M uniformly on compact time intervals. Therefore, $(Y_{Nt}^N)_{t \in \mathbb{R}_+}$ converges in distribution to M .

For our approach to work one needs to make sure that for every $N \in \mathbb{N}$ and $y \in (l, r)$ there exists a scale factor $a_N(y)$ such that the distribution of $y + a_N(y)X_i$ can be embedded into the diffusion M , conditioned to $M_0 = y$, with a stopping time with expectation $1/N$. The collection of distributions that can be embedded into M with integrable stopping times is fully described in [1]. Moreover, there is a closed form integral expression for the minimal expectation of an embedding stopping time (see Theorem 3 in [1]). This allows us to derive weak sufficient conditions (see Section 2) for the existence of a scale factor $a_N: (l, r) \rightarrow (0, \infty)$ such that (Y_k^N) can be embedded into M with stopping times having expectation $1/N$.

Our approach to generalize Donsker's theorem is essentially different from the one pioneered by Stone in [12] (also see [2] for a recent generalization to tree-valued processes). In that approach the approximating processes are *continuous-time* Markov processes that *do not jump over points in their state spaces* (that is, they can be e.g. diffusions or birth and death processes). On the contrary, in this paper we approximate M via *discrete-time* Markov chains. Another conceptual difference is that we develop our theory without requiring that the approximating Markov chains do not jump over points. On an informal level, one might view conditions (18)–(19) and (26)–(29) at which we arrive in Section 2 as an indication of what comes out when we want to allow overjumping.

The paper is organized in the following way. In Section 1, we recall a necessary and sufficient condition, derived in [1], for a distribution to be embeddable into the diffusion M with an *integrable* stopping time. Moreover, we slightly generalize an integral formula for the minimal expectation of an embedding stopping time. In Section 2, we characterize families of scaled random walks whose laws can be embedded into M via a sequence of increasing stopping times such that the expected distance between two consecutive stopping times is equal to $1/N$, for $N \in \mathbb{N}$. In Section 3, we provide sufficient conditions for a sequence of scaled random walks, embeddable into M , to converge in distribution to M .

1. Embedding distributions in integrable time

In this section we recall a necessary and sufficient condition from [1] for a centered distribution to be embeddable in a diffusion via an integrable stopping time.

Let $I = (l, r)$ with $l \in [-\infty, \infty)$ and $r \in (-\infty, \infty]$. As usual we denote by \bar{I} the

closure of I in \mathbb{R} . Let $\eta : \mathbb{R} \rightarrow \mathbb{R}$ be a Borel-measurable function satisfying

$$\eta(x) \neq 0 \text{ for all } x \in I, \quad (3)$$

$$\frac{1}{\eta^2} \in L^1_{\text{loc}}(I), \quad (4)$$

$$\eta(x) = 0 \text{ for all } x \in \mathbb{R} \setminus I, \quad (5)$$

where $L^1_{\text{loc}}(I)$ denotes the set of functions that are locally integrable on I .

Consider the SDE

$$dM_t = \eta(M_t)dW_t, \quad M_0 \sim \gamma, \quad (6)$$

where γ is a probability measure on I . The assumptions (3)–(5) imply that (6) possesses a weak solution that is unique in law (see e.g. [5] or Theorem 5.5.7 in [7]). This means that there exists a pair of processes (M, W) on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, with (\mathcal{F}_t) satisfying the usual conditions, such that W is an (\mathcal{F}_t) -Brownian motion, M_0 is an \mathcal{F}_0 -measurable random variable with distribution γ and (M, W) satisfies the SDE (6). Let us note that M stays in l (resp. r) once it hits l (resp. r).

For all $y \in I$ and $x \in \mathbb{R}$ we define

$$q(y, x) = \int_y^x \int_y^u \frac{2}{\eta^2(z)} dz du.$$

Notice that Itô's formula implies that the process $(q(M_0, M_t) - t)$ is a local martingale starting in 0. The assumptions (3)–(5) imply that for all $y \in I$ the nonnegative function $q(y, \cdot)$ is finite on I and equal to ∞ on $\mathbb{R} \setminus [l, r]$. Besides, $q(y, \cdot)$ is strictly convex on I , strictly decreasing to zero on (l, y) and strictly increasing from zero on (y, r) . Moreover, for all $y, \bar{y} \in I$ and $x \in \mathbb{R}$ we have

$$q(y, x) = q(\bar{y}, x) - q(\bar{y}, y) - q_x(\bar{y}, y)(x - y), \quad (7)$$

where q_x denotes the partial derivative of q with respect to the second argument.

Recall that by Feller's test for explosions we have $q(y, l+) = \infty$ if and only if the probability for the process M to attain l in finite time is equal to zero. Notice that the non-explosion condition $q(y, l+) = \infty$ does *not* depend on y . Moreover, if $l = -\infty$, then $q(y, l+) = \infty$, and hence any solution of (6) does not attain $-\infty$ in finite time. Similar statements hold true for the right-hand side boundary r .

We next recall a result from [1] providing a necessary and sufficient condition for a distribution to be embeddable in M with an integrable stopping time. Let μ be a centered probability measure on \mathbb{R} , i.e. $\int |x| \mu(dx) < \infty$ and $\int x \mu(dx) = 0$. Moreover, we assume that $\mu \neq \delta_0$. Let $K(y, a, \cdot)$, $y \in \bar{I}$, $a \in \mathbb{R}_+$, be the location-scale family of probability distributions defined by

$$K(y, a, B) = \mu\left(\frac{B - y}{a}\right), \quad B \in \mathcal{B}(\mathbb{R}), \quad (8)$$

whenever $a > 0$; and $K(y, 0, \cdot) = \delta_y$. Consider the problem of finding a stopping time τ such that

$$\text{Law}(M_\tau | \mathcal{F}_0) = K(M_0, a(M_0), \cdot), \quad (9)$$

where $\text{Law}(M_\tau | \mathcal{F}_0)$ denotes the regular conditional distribution of M_τ with respect to \mathcal{F}_0 and $a: \bar{I} \rightarrow \mathbb{R}_+$ is a given Borel function. The unconditional version of this problem is usually referred to as the *Skorokhod embedding problem* or the *SEP*, see [6] or [9] for an overview. In the subsequent sections we need embedding stopping times that are integrable conditionally on \mathcal{F}_0 , i.e. that satisfy $E[\tau | \mathcal{F}_0] < \infty$ a.s. For all $y \in I$ we define

$$Q(y) = \int_{\mathbb{R}} q(y, x) K(y, a(y), dx). \quad (10)$$

One can show that $Q(y)$ is the minimal expected time required for embedding $K(y, a(y), \cdot)$ into M , conditional to $M_0 = y$. To provide an intuition, suppose that the starting in 0 local martingale $(q(M_0, M_t) - t)$ is a true martingale and τ is a solution of the embedding problem (9). If the optional sampling theorem applies, then $E[\tau | \mathcal{F}_0] = E[q(M_0, M_\tau) | \mathcal{F}_0] = Q(M_0)$. More formally, we have the following result, which is a straightforward generalization of Theorem 3 and Proposition 4 in [1]:

Theorem 1.1. (i) Any (\mathcal{F}_t) -stopping time τ solving (9) satisfies $E[\tau | \mathcal{F}_0] \geq Q(M_0)$ a.s.
(ii) There exists a solution τ of the embedding problem (9) satisfying the property $E[\tau | \mathcal{F}_0] < \infty$ a.s. if and only if

$$Q(M_0) < \infty \quad \text{a.s.} \quad (11)$$

In this case, there exists an embedding stopping time τ with

$$E[\tau | \mathcal{F}_0] = Q(M_0) \quad \text{a.s.} \quad (12)$$

For the proof of the main results of Section 3 it turns out to be helpful to work with the specific solution of the embedding problem (9) provided in [1]. For the reader's convenience we briefly explain the solution method in the Appendix.

2. Embedding scaled random walks

Let (M, W) be a weak solution of

$$dM_t = \eta(M_t) dW_t, \quad M_0 = m, \quad (13)$$

with $m \in I$. Moreover, let X_1, X_2, \dots be a sequence of i.i.d. real-valued integrable random variables with $E(X_i) = 0$. We denote the distribution of X_i by μ . Throughout we assume that $\mu \neq \delta_0$.

Definition 2.1. Let $a : \mathbb{R} \rightarrow \mathbb{R}_+$ be a Borel function. The process $Y = (Y_k)_{k \in \mathbb{Z}_+}$, defined by $Y_0 = m$ and

$$Y_{k+1} = Y_k + a(Y_k)X_{k+1}, \quad k \geq 0, \quad (14)$$

is called *random walk generated by (X_k) with scale factor a and starting point m* .

We say that $Y = (Y_k)_{k \in \mathbb{Z}_+}$ is a *scaled random walk* if there exists a scale factor a such that Y is the random walk generated by (X_k) with scale factor a .

In this section we aim at constructing scale factors such that (Y_k) can be embedded in distribution into M with a sequence of stopping times (τ_k) such that $(M_{\tau_k}) \stackrel{d}{=} (Y_k)$, that is, both discrete time processes have the same law. More precisely, we solve the following problem.

Problem (P). Let $N \in \mathbb{N}$. Does there exist a scale factor a_N such that the associated scaled random walk $(Y_k^N)_{k \in \mathbb{Z}_+}$ with $Y_0^N = m$ can be embedded in distribution into M with a sequence of (\mathcal{F}_t) -stopping times $(\tau_k^N)_{k \in \mathbb{Z}_+}$ with

$$\tau_0^N = 0 \quad \text{and} \quad E[\tau_{k+1}^N | \mathcal{F}_{\tau_k^N}] = \tau_k^N + \frac{1}{N}, \quad (15)$$

for all $k \geq 0$?

In order to determine the scale factor solving Problem (P), we introduce, for all $y \in I$, the mapping $G_y : [0, \infty) \rightarrow [0, \infty]$ defined via

$$G_y(a) = \int_{\mathbb{R}} q(y, x) K(y, a, dx) = \int_{\mathbb{R}} q(y, y + ax) \mu(dx). \quad (16)$$

Recall that $G_y(a)$ is the minimal expected time needed for embedding $K(y, a, dx)$ into M (cf. Theorem 1.1 and the discussion following (10)). Notice that, for all $y \in I$, the map $G_y(\cdot)$ is strictly increasing with $G_y(0) = 0$, left-continuous by the monotone convergence theorem, and continuous on $[0, \infty) \setminus \{a_{\inf}\}$ with

$$a_{\inf} := \inf\{a \in [0, \infty) : G_y(a) = \infty\}, \quad \inf \emptyset := \infty,$$

by the dominated convergence theorem.

We now provide sufficient conditions guaranteeing that a solution of Problem (P) exists. We need to distinguish four cases.

2.1. Case 1: $l = -\infty$ and $r = \infty$

In this subsection we make the following assumption.

(A1) There exists $y \in I$ such that $G_y(a) < \infty$ for all $a > 0$.

Lemma 2.2. (A1) is equivalent to the condition that for all $y \in I$ and $a > 0$ we have $G_y(a) < \infty$.

Proof. Let $\bar{y} \in I$ and suppose that $G_{\bar{y}}(a) < \infty$ for all $a > 0$. Let $y \in I$ and notice that $q(y, x) = q(\bar{y}, x) - q(\bar{y}, y) - q_x(\bar{y}, y)(x - y)$. Since μ is centered, we have

$$G_y(a) = \int_{\mathbb{R}} q(y, y + ax) \mu(dx) = \int_{\mathbb{R}} q(\bar{y}, y + ax) \mu(dx) - q(\bar{y}, y).$$

For all $a > 0$ and $x \in \mathbb{R}$ with $|x| \geq \frac{|y - \bar{y}|}{a}$ we have

$$q(\bar{y}, y + ax) \leq q(\bar{y}, \bar{y} + 2ax).$$

From this we obtain $G_y(a) < \infty$. □

The following theorem provides a solution to Problem (P) in Case 1.

Theorem 2.3. *If (A1) is satisfied, then for all $N \in \mathbb{N}$ there exists a unique scale factor a_N satisfying*

$$G_y(a_N(y)) = \frac{1}{N}, \quad y \in I. \quad (17)$$

Moreover, the random walk $(Y_k^N)_{k \in \mathbb{Z}_+}$ generated by (X_k) with scale factor a_N and starting point m can be embedded into M with a sequence of stopping times satisfying (15).

Remark 2.4. It is worth noting that (A1) is satisfied whenever μ has compact support.

For the proof of the theorem we need the following auxiliary result.

Lemma 2.5. *If (A1) is satisfied, then G_y is a bijective mapping from $[0, \infty)$ to $[0, \infty)$, for all $y \in I$.*

Proof. Notice that $\lim_{x \rightarrow \pm\infty} q(y, x + y) = \infty$. Moreover, if $0 \leq a < b$ and $x \neq 0$, then $q(y, y + ax) < q(y, y + bx)$. Therefore, G_y is strictly increasing and by monotone convergence, $\lim_{a \rightarrow \infty} G_y(a) = \infty$. Condition (A1), Lemma 2.2 and a dominated convergence argument show that G_y is continuous, and consequently, bijective. □

Proof of Theorem 2.3. Let $N \in \mathbb{N}$. Lemma 2.5 implies that for all $y \in I$ there exists a scale factor $a_N(y)$ that satisfies (17).

We next define a sequence of stopping times $(\tau(k))_{k \in \mathbb{Z}_+}$ that embeds the transition probabilities into the diffusion M . First define $\tau(0) = 0$. Suppose that $\tau(k)$ is already defined. Set $X_t = M_{t + \tau(k)}$, for $t \geq 0$, and observe that

$$dX_t = \eta(X_t) d(W_{t + \tau(k)} - W_{\tau(k)}), \quad X_0 = M_{\tau(k)}.$$

Theorem 1.1 implies that there exists an $(\mathcal{F}_{t + \tau(k)})$ -stopping time $\rho(k + 1)$ with

$$E[\rho(k + 1) | \mathcal{F}_{\tau(k)}] = Q(M_{\tau(k)}) = G_{M_{\tau(k)}}(a_N(M_{\tau(k)})) = \frac{1}{N}$$

such that $X_{\rho(k+1)} \stackrel{d}{=} Y_k^N + a_N(Y_k^N)X_{k+1}$. Now define $\tau(k + 1) = \tau(k) + \rho(k + 1)$. By construction, the sequence $(M_{\tau(k)})_{k \in \mathbb{Z}_+}$ has the same distribution as $(Y_k^N)_{k \in \mathbb{Z}_+}$. □

The next example shows that a scale factor satisfying (17) does not necessarily exist if (A1) does not hold true.

Example 2.6. Let μ be the probability measure with density $f(x) = c \frac{e^{-|x|}}{1+x^2}$, where $c = \left(\int \frac{e^{-|x|}}{1+x^2} dx \right)^{-1}$. Moreover let $m = 0$ and $\eta(x) = e^{-x/2}$, $x \in \mathbb{R}$. Then we have $q(y, x) = 2e^y(e^{x-y} - (x-y) - 1)$. A straightforward calculation shows that $G_y(a) = \infty$ for $a > 1$. Therefore Condition (A1) is not satisfied. Moreover, for $a = 1$ we have that

$$G_y(1) = 2e^y c \int_{\mathbb{R}} (e^x - x - 1) \frac{e^{-|x|}}{1+x^2} dx < \infty.$$

By considering the limit $y \rightarrow -\infty$ we see that for every $N \in \mathbb{N}$ there exists $y \in \mathbb{R}$ such that $G_y(1) < 1/N$. In particular, there exists no solution to (17).

2.2. Case 2: $l > -\infty$ and $r = \infty$

Here we impose the following assumption.

(A2) $\inf \text{supp } \mu > -\infty$ and there exists $y \in I$ such that the integral over the positive real line $\int_{\mathbb{R}_+} q(y, y+ax) \mu(dx) < \infty$ for all $a > 0$.

For every $y > l$ we set $\bar{a}(y) = \frac{l-y}{\inf \text{supp } \mu}$. Note that for all $a \leq \bar{a}(y)$ we have $a \text{supp } \mu \subset [l-y, \infty)$. In the following we use the short-hand notation $q(x) = q(m, x)$, $x \in \mathbb{R}$.

We now present a solution to Problem (P) in Case 2.

Theorem 2.7. *Suppose that (A2) is satisfied and additionally that the following implications hold true:*

$$\text{if } q(l+) < \infty, \quad \text{then } \mu(\{\inf \text{supp } \mu\}) > 0, \quad (18)$$

$$\text{if } q(l+) = \infty, \quad \text{then } \liminf_{y \rightarrow \infty} G_y(\bar{a}(y)) > 0 \text{ and } \liminf_{y \searrow l} G_y(\bar{a}(y)) > 0. \quad (19)$$

Then there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ there exists a unique scale factor a_N satisfying

$$a_N(y) = \sup \left\{ a \in [0, \infty) : G_y(a) \leq \frac{1}{N} \right\}, \quad y \in I, \quad (20)$$

and $a_N(l) = 0$. Moreover, the random walk $(Y_k^N)_{k \in \mathbb{Z}_+}$, generated by (X_k) with scale factor a_N and starting in m , can be embedded in M with stopping times satisfying (15).

In the case $q(l+) < \infty$, we can take $N_0 = 1$; while in the case $q(l+) = \infty$, the scale factor a_N of (20) satisfies (17) for all $y \in I$ and $N \geq N_0$ (here $N_0 \geq 1$ can be necessary).

Remark 2.8. It is worth noting that the assumptions of Theorem 2.7 are satisfied whenever μ has compact support and $\mu(\{\inf \text{supp } \mu\}) > 0$ (see Proposition 2.12).

Proof. From similar arguments as in the proof of Lemma 2.2 it follows that condition (A2) implies $\int_{\mathbb{R}_+} q(y, y + ax) \mu(dx) < \infty$ for all $y \in I$ and $a > 0$. Notice that the sup in (20) is attained, since G_y is left-continuous. As in the proof of Lemma 2.5 one can show that $G_y : [0, \bar{a}(y)] \rightarrow [0, G_y(\bar{a}(y))]$ is bijective.

Now assume $q(l+) < \infty$. (18) implies that $w = \mu(\{\inf \text{supp } \mu\})$ is positive. Let $N \in \mathbb{N}$ and $\nu_N(y, B) = K(y, a_N(y), B)$ for $y \in I$ and $B \in \mathcal{B}(\mathbb{R})$. Similarly to the proof of Theorem 2.3 we construct a sequence of stopping times $(\tau(k))_{k \in \mathbb{Z}_+}$ embedding the transition probabilities into M . Let $\tau(0) = 0$. Suppose that $\tau(k)$ is defined. By Theorem 1.1 there exists a stopping time $\rho(k+1)$ that embeds $\nu_N(M_{\tau(k)}, \cdot)$ into the process $X_t = M_{t+\tau(k)}$, $t \geq 0$, and satisfies

$$E[\rho(k+1)|\mathcal{F}_{\tau(k)}] = Q(M_{\tau(k)}) = G_{M_{\tau(k)}}(a_N(M_{\tau(k)})) \leq \frac{1}{N}.$$

Define $\tau(k+1) = \tau(k) + \rho(k+1)$ if $M_{\tau(k)+\rho(k+1)} > l$, and $\tau(k+1) = \tau(k) + \rho(k+1) + \frac{1}{w}[\frac{1}{N} - Q(M_{\tau(k)})]$ if $M_{\tau(k)+\rho(k+1)} = l$. Then we have

$$\begin{aligned} E[\tau(k+1)|\mathcal{F}_{\tau(k)}] &= \tau(k) + Q(M_{\tau(k)}) + \frac{1}{w} \left[\frac{1}{N} - Q(M_{\tau(k)}) \right] P(M_{\tau(k)+\rho(k+1)} = l | \mathcal{F}_{\tau(k)}) \\ &= \tau(k) + \frac{1}{N}. \end{aligned}$$

Next assume that $q(l+) = \infty$. Due to (19) and Lemma 2.9 below, we have

$$\inf_{y \in I} G_y(\bar{a}(y)) > 0.$$

Choosing $N_0 \in \mathbb{N}$ such that $1/N_0 < \inf_{y \in I} G_y(\bar{a}(y))$ yields that for every $N \geq N_0$ and $y \in I$ we have $G_y(a_N(y)) = \frac{1}{N}$. The rest of the proof goes along the lines of the proof of Theorem 2.3. \square

Lemma 2.9. *Suppose (A2).*

- (i) *The function $y \mapsto G_y(\bar{a}(y))$ is a lower semicontinuous function $I \rightarrow (0, \infty]$.*
- (ii) *For any compact subinterval $J \subset I$, we have*

$$\inf_{y \in J} G_y(\bar{a}(y)) > 0.$$

Proof. To simplify notation we assume that $\inf \text{supp } \mu = -1$. For $y > l$, we have $G_y(\bar{a}(y)) > 0$, $\bar{a}(y) = y - l$ and $g(y) := G_y(\bar{a}(y)) = \int q(y, y + (y - l)x) \mu(dx)$. Since $q(y, \cdot)$ nonnegative, Fatou's lemma yields for $y_0 \in I$

$$\liminf_{y \rightarrow y_0} g(y) \geq \int \liminf_{y \rightarrow y_0} q(y, y + (y - l)x) \mu(dx) = g(y_0). \quad (21)$$

This proves the first statement. The second statement immediately follows from the first one. \square

Remark 2.10. In comparison with Case 1 the condition that determines the scale factor changes from (17) to a less pleasant one (20). Notice that in Case 1, conditions (17) and (20) are equivalent. As stated in Theorem 2.7, in Case 2 with $q(l+) = \infty$ again (17) holds true. Example 2.11 below shows that, in Case 2 with $q(l+) < \infty$, it can indeed happen that the scale factor does not satisfy (17) any longer. However, by Lemma 2.9 (ii), the following statement holds:

(Eq) Suppose (A2) and, for $N \in \mathbb{N}$, consider the scale factor a_N satisfying (20). Then, for any compact subinterval $J \subset I$, there exists $N_1 \in \mathbb{N}$ such that, for all $N \geq N_1$, the scale factor a_N satisfies (17) on J .

Example 2.11. Let M be a Brownian motion starting at $m > 0$ and absorbed as it reaches zero, i.e. we have $I = (0, \infty)$ and $\eta \equiv 1$. Let $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$. A short computation shows that, for $y > 0$,

$$G_y(a) = \begin{cases} a^2 & \text{if } a \in [0, y], \\ \infty & \text{if } a \in (y, \infty). \end{cases}$$

Hence, $a_N(y) = \frac{1}{\sqrt{N}} \wedge y$, and (17) fails whenever $y \in (0, \frac{1}{\sqrt{N}})$.

While (18) is a condition on the primitives of our problem, (19) is harder to verify. In the sequel we present sufficient conditions on μ (Proposition 2.12) and η (Proposition 2.13) that imply (19).

Proposition 2.12. *Suppose (A2). If $\mu(\{\inf \text{supp } \mu\}) > 0$, then (19) is satisfied. Moreover, Theorem 2.7 applies with $N_0 = 1$.*

Proof. From $q(l+) = \infty$ and $\mu(\{\inf \text{supp } \mu\}) > 0$ it follows that $G_y(\bar{a}(y)) = \infty$ for all $y \in I$, which implies the claims. \square

Proposition 2.13. *Under (A2) assume that*

$$\limsup_{x \searrow l} \frac{|\eta(x)|}{x-l} < \infty \quad \text{and} \quad \limsup_{x \nearrow \infty} \frac{|\eta(x)|}{x} < \infty.$$

Then (19) is satisfied.

Proof. To simplify notation we assume that $\inf \text{supp } \mu = -1$. For $y > l$ we have $\bar{a}(y) = y - l$ and $g(y) := G_y(\bar{a}(y)) = \int h(x, y) \mu(dx)$ with $h(x, y) := q(y, y + (y - l)x)$. We need to show that $\liminf_{y \rightarrow y_0} g(y) > 0$ for $y_0 \in \{l, \infty\}$. Note that

$$h(x, y) = 2 \int_0^x \int_0^u \frac{(y-l)^2}{\eta^2((y-l)z + y)} dz du. \quad (22)$$

We have for every $z > -1$

$$\liminf_{y \searrow l} \left(\frac{y-l}{|\eta((y-l)z + y)|} \right) = \frac{1}{z+1} \liminf_{y \searrow l} \left(\frac{(y-l)z + y-l}{|\eta((y-l)z + y)|} \right) = \frac{1}{z+1} \liminf_{x \searrow l} \left(\frac{x-l}{|\eta(x)|} \right) > 0$$

and

$$\liminf_{y \nearrow \infty} \left(\frac{y-l}{|\eta((y-l)z+y)|} \right) = \frac{1}{z+1} \liminf_{x \nearrow \infty} \left(\frac{x-l}{|\eta(x)|} \right) = \frac{1}{z+1} \liminf_{x \nearrow \infty} \left(\frac{x}{|\eta(x)|} \right) > 0.$$

Thus, for $y_0 \in \{l, \infty\}$, applying Fatou's lemma in (22) (observe that the area of integration is positively oriented also for $x \leq 0$) yields $\liminf_{y \rightarrow y_0} h(x, y) > 0$ for every $x \in (-1, \infty) \setminus \{0\}$. Now the argument similar to (21) yields the claim. \square

Remark 2.14. We can replace the conditions $\limsup_{x \searrow l} \frac{|\eta(x)|}{x-l} < \infty$ and $\limsup_{x \nearrow \infty} \frac{|\eta(x)|}{x} < \infty$ in the formulation of Proposition 2.13 by the weaker conditions $\liminf_{y \searrow l} h(\cdot, y) > 0$ and $\liminf_{y \nearrow \infty} h(\cdot, y) > 0$ on a set of positive mass with respect to μ , where h is defined as in the proof of Proposition 2.13.

Let us illustrate in more detail how the assumptions in Theorem 2.7 work when $q(l+) = \infty$. Recall that, in the case $q(l+) = \infty$, the scale factor a_N satisfies (17) (not only (20)). If, however, (A2) does not hold true, then a scale factor satisfying (17) does not necessarily exist. This can be shown by means of an example similar to Example 2.6. The role of condition (19) is as follows. Together with (A2) it guarantees that, in the case $q(l+) = \infty$, there is a scale factor satisfying (17). Examples 2.15 and 2.16 below show that (19) can fail and a scale factor satisfying (17) does not necessarily exist when we require (A2) alone.

Example 2.15. Let us consider the constant elasticity of variance (CEV) process with $\alpha > 1$, i.e. $l = 0$, $r = \infty$ and $\eta(x) = x^\alpha$ on I . For $y > 0$, we have

$$q(y, x) = \begin{cases} \frac{2}{2\alpha-1} \left[\frac{1}{2\alpha-2} \left(\frac{1}{x^{2\alpha-2}} - \frac{1}{y^{2\alpha-2}} \right) + \frac{x-y}{y^{2\alpha-1}} \right] & \text{if } x \geq 0, \\ \infty & \text{if } x < 0, \end{cases} \quad (23)$$

in particular, $q(y, 0+) = \infty$. We see that any centered measure $\mu \neq \delta_0$ with $\inf \text{supp } \mu > -\infty$ satisfies (A2).

Notice that $q(y/a, x/a) = a^{2\alpha-2} q(y, x)$ for $a, x, y > 0$. With $b = -\inf \text{supp } \mu > 0$ we now calculate $\bar{a}(y) = y/b$ and

$$G_y(\bar{a}(y)) = \frac{1}{y^{2\alpha-2}} \int_{\mathbb{R}} q\left(1, 1 + \frac{x}{b}\right) \mu(dx) = \frac{b^{2\alpha-2}}{y^{2\alpha-2}} \int_{\mathbb{R}} q(b, b+x) \mu(dx).$$

Thus, Theorem 2.7 applies if and only if $\mu \neq \delta_0$ is centered, $\inf \text{supp } \mu > -\infty$ and, for some $\varepsilon > 0$,

$$\int_{[\inf \text{supp } \mu, \inf \text{supp } \mu + \varepsilon]} q(-\inf \text{supp } \mu, -\inf \text{supp } \mu + x) \mu(dx) = \infty. \quad (24)$$

Here we used that such an integral over \mathbb{R} is infinite if and only if (24) is satisfied (for any $y > 0$, the function $x \mapsto q(y, x)$ has linear growth as $x \rightarrow \infty$). Notice that a sufficient condition for (24) is $\mu(\{\inf \text{supp } \mu\}) > 0$.

The previous example shows that, choosing a centered measure $\mu \neq \delta_0$ with $\inf \text{supp } \mu > -\infty$ in a way that (24) fails, we have (A2) but violate (19) in the way $G_y(\bar{a}(y)) \rightarrow 0$ as $y \rightarrow \infty$. This raises the question of whether it is possible to violate (19), under (A2), in the way $G_y(\bar{a}(y)) \rightarrow 0$ as $y \searrow l$. This must be more delicate because, on the one hand, the condition $|\eta(x)| \geq c(x-l)^\alpha$, for all $x \in (l, b)$, with some $c > 0$, $b > l$ and $\alpha < 1$, implies $q(l+) < \infty$, while, on the other hand, the condition $|\eta(x)| \leq c(x-l)$, for all $x \in (l, b)$, with some $c > 0$ and $b > l$, implies $\liminf_{y \searrow l} G_y(\bar{a}(y)) > 0$ by Proposition 2.13. Still this is possible as the following example shows.

Example 2.16. We consider again $I = (0, \infty)$ and define

$$\eta(x) = \begin{cases} 2\sqrt{2}x \frac{(-\log x)^{\frac{3}{4}}}{\sqrt{-1-2\log x}} & \text{if } x \in (0, 1/2), \\ 1 & \text{if } x \in [1/2, \infty). \end{cases}$$

Then one can verify that

$$q(1/2, x) = \begin{cases} \sqrt{-\log x} + \frac{1}{\sqrt{\log 2}} \left(x - \frac{1}{2}\right) - \sqrt{\log 2} & \text{if } x \in (0, 1/2), \\ x^2 - x + 1/4 & \text{if } x \in [1/2, \infty), \end{cases}$$

in particular, $q(1/2, 0+) = \infty$. We see that any centered measure $\mu \neq \delta_0$ with $\inf \text{supp } \mu > -\infty$ and $\int_{\mathbb{R}_+} x^2 \mu(dx) < \infty$ satisfies (A2).

Let now μ be a centered measure with

$$\inf \text{supp } \mu = -1 \text{ and } \sup \text{supp } \mu = 1,$$

in particular, $\bar{a}(y) = y$ for $y > 0$. Moreover assume that

$$\int_{\mathbb{R}} -\log(x+1) \mu(dx) < \infty. \quad (25)$$

By formula (7), we have for $x \in (-1, 1]$ and $y \in (0, 1/4]$

$$q(y, y+xy) = \sqrt{-\log[y(x+1)]} - \sqrt{-\log y} + \frac{x}{2\sqrt{-\log y}}.$$

In particular, for any $x \in (-1, 1]$, the mapping $y \mapsto q(y, y+xy)$ is increasing with $q(y, y+xy) \rightarrow 0$ as $y \searrow 0$. Indeed, we have

$$\sqrt{-\log[y(x+1)]} - \sqrt{-\log y} = \frac{-\log(x+1)}{\sqrt{-\log[y(x+1)]} + \sqrt{-\log y}}.$$

Then dominated convergence (cf. (25)) ensures that $G_y(\bar{a}(y)) \rightarrow 0$ as $y \searrow 0$.

Finally, we illustrate how Theorem 2.7 works when $q(l+) < \infty$.

Example 2.17. Let us now consider the CEV process with $\alpha \in (-\infty, 1) \setminus \{0\}$, i.e. $I = (0, \infty)$ and $\eta(x) = x^\alpha$ on I . In the case $\alpha \neq \frac{1}{2}$, for $y > 0$, the function $q(y, \cdot)$ is given by formula (23). In the case $\alpha = \frac{1}{2}$, for $y > 0$, we have

$$q(y, x) = \begin{cases} 2x \log \frac{x}{y} - 2(x - y) & \text{if } x \geq 0, \\ \infty & \text{if } x < 0. \end{cases}$$

In particular, $q(y, 0+) < \infty$ in both cases. Thus, Theorem 2.7 applies if and only if $\mu \neq \delta_0$ is centered, $\inf \text{supp } \mu > -\infty$, $\mu(\{\inf \text{supp } \mu\}) > 0$ and

$$\begin{aligned} \text{if } \alpha = \frac{1}{2}, \quad & \text{then } \int_{\mathbb{R}_+} x \log x \mu(dx) < \infty, \\ \text{if } \alpha < \frac{1}{2}, \quad & \text{then } \int_{\mathbb{R}_+} x^{2-2\alpha} \mu(dx) < \infty. \end{aligned}$$

2.3. Case 3: $l = -\infty$ and $r < \infty$

This case can be reduced to Case 2 by considering the diffusion $-M$.

2.4. Case 4: $l > -\infty$ and $r < \infty$

In this subsection we make the following assumption.

(A3) $\inf \text{supp } \mu > -\infty$ and $\sup \text{supp } \mu < \infty$.

For every $y \in I$ we set $\bar{a}(y) = \frac{l-y}{\inf \text{supp } \mu} \wedge \frac{r-y}{\sup \text{supp } \mu}$. Note that for all $a \leq \bar{a}(y)$ we have $a \text{supp } \mu \subset [l-y, r-y]$.

A solution to Problem (P) in Case 4 is given in the next theorem.

Theorem 2.18. *Suppose that (A3) is satisfied and additionally that the following implications hold true:*

$$\text{if } q(l+) < \infty, \quad \text{then } \mu(\{\inf \text{supp } \mu\}) > 0, \quad (26)$$

$$\text{if } q(l+) = \infty, \quad \text{then } \liminf_{y \searrow l} G_y(\bar{a}(y)) > 0, \quad (27)$$

$$\text{if } q(r-) < \infty, \quad \text{then } \mu(\{\sup \text{supp } \mu\}) > 0, \quad (28)$$

$$\text{if } q(r-) = \infty, \quad \text{then } \liminf_{y \nearrow r} G_y(\bar{a}(y)) > 0. \quad (29)$$

Then there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ there exists a unique scale factor a_N satisfying (20) and $a_N(l) = a_N(r) = 0$. Moreover, the random walk (Y_k^N) , scaled with a_N and starting in m , is embeddable in M with stopping times $(\tau_k^N)_{k \in \mathbb{Z}_+}$ satisfying (15).

Proof. Similar to the proof of Theorem 2.7. \square

Remark 2.19. Notice that the assumptions of Theorem 2.18 are satisfied whenever μ has compact support, $\mu(\{\inf \text{supp } \mu\}) > 0$ and $\mu(\{\sup \text{supp } \mu\}) > 0$ (see Proposition 2.20).

The next two propositions provide sufficient conditions for the properties (27) and (29) to hold true.

Proposition 2.20. *Suppose (A3). If $\mu(\{\inf \text{supp } \mu\}) > 0$, then (27) is satisfied.*

Proof. Similar to the proof of Proposition 2.12. \square

Similarly, the condition $\mu(\{\sup \text{supp } \mu\}) > 0$ is sufficient for (29).

Proposition 2.21. *Suppose (A3). If $\limsup_{x \searrow l} \frac{|\eta(x)|}{x-l} < \infty$, then (27) is satisfied. If $\limsup_{x \nearrow r} \frac{|\eta(x)|}{r-x} < \infty$, then (29) is satisfied.*

Proof. Similar to the proof of Proposition 2.13. \square

Finally, it is worth noting that the detailed discussions in Case 2 about the role of different assumptions, etc., have their analogues in Case 4. In particular, we have:

- The statements of Lemma 2.9 apply verbatim under (A3) instead of (A2);
- Statement (Eq) in Remark 2.10 applies verbatim under (A3) instead of (A2);
- The conclusion of Theorem 2.18 holds true with $N_0 = 1$ whenever (A3) is satisfied and we have

$$\mu(\{\inf \text{supp } \mu\}) > 0 \quad \text{and} \quad \mu(\{\sup \text{supp } \mu\}) > 0$$

(cf. with the statements in the end of Theorem 2.7 and Proposition 2.12);

- Under the assumptions of Theorem 2.18, for all $N \geq N_0$ and $y \in I$, the scale factors a_N satisfy (17) (not only (20)) whenever $q(l+) = q(r-) = \infty$ (cf. with the statement in the end of Theorem 2.7).

3. Weak convergence

In this section we use the setting and notations of Section 2. In particular, we consider a weak solution (M, W) of (13), denote by (Y_k^N) the scaled random walk (14), and assume that $\mu \neq \delta_0$ is a centered probability measure on \mathbb{R} . Throughout this section we suppose that one of the sufficient conditions from Section 2 is satisfied that guarantees, for sufficiently large $N \in \mathbb{N}$, the existence of a scale factor a_N satisfying (20) that solves Problem (P). Let us remark that under each of these sufficient conditions, we have

$$\int_{\mathbb{R}} q(m + ax) \mu(dx) < \infty \quad \text{for some } a > 0 \tag{30}$$

(recall that $q(x)$ is a short-hand notation for $q(m, x)$). We extend Y^N to a continuous-time process on \mathbb{R}_+ via linear interpolation, i.e. for all $t \geq 0$ we set $Y_t^N = Y_{[t]}^N + (t - [t])(Y_{[t]+1}^N - Y_{[t]}^N)$.

In this section we show that if the diffusion coefficient η is locally bounded away from 0 and from $\pm\infty$ and μ has compact support, then the sequence of continuous processes

$(Y_{Nt}^N)_{t \geq 0}$ converges in law to the process $(M_t)_{t \geq 0}$, as $N \rightarrow \infty$ (see Theorem 3.6). We also present other sets of sufficient conditions for this weak convergence (generally, the less we require on η , the more we need to require on μ). One can thus interpret $(Y_k^N)_{k \in \mathbb{Z}_+}$ as a Markov chain approximating the diffusion $(M_t)_{t \geq 0}$.

To simplify the analysis, we only show the weak convergence on the time interval $[0, 1]$. A straightforward generalization implies the weak convergence on \mathbb{R}_+ .

We first assume that η satisfies the following condition.

(C1) $|\eta|$ and $\frac{1}{|\eta|}$ are bounded on I .

Theorem 3.1. *Suppose that (C1) holds true. Then the processes $(Y_{Nt}^N)_{t \in [0, 1]}$ converge to $(M_t)_{t \in [0, 1]}$ in distribution, as $N \rightarrow \infty$, i.e. the associated measures on $(C[0, 1], \mathcal{B}(C[0, 1]))$ converge weakly.*

We first show that boundedness of $|\eta|$ implies that the scale factor $a_N(y)$ is of order $\frac{1}{\sqrt{N}}$.

Lemma 3.2. *If $|\eta(x)| \leq U < \infty$ for all $x \in I$, then there exists $A \in \mathbb{R}_+$ such that $a_N(y) \leq \frac{A}{\sqrt{N}}$ for all $N \in \mathbb{N}$ and $y \in I$.*

Proof. For $x, y \in I$ we have

$$q(y, x) \geq \int_y^x \int_y^u \frac{2}{U^2} dz du = \frac{(x - y)^2}{U^2}. \quad (31)$$

In particular, $q(x) \geq (x - m)^2/U^2$, and hence (30) implies that $\int x^2 \mu(dx) < \infty$. It follows from (20) and (31) that

$$\frac{1}{N} \geq G_y(a_N(y)) = \int q(y, y + a_N(y)x) \mu(dx) \geq \frac{a_N^2(y) \int x^2 \mu(dx)}{U^2},$$

which yields the claim with $A = \frac{U}{\sqrt{\int x^2 \mu(dx)}}$. □

Lemma 3.3. *Assume (C1). The solution to Problem (P) can be chosen in such a way that the $(\mathcal{F}_{\tau^N(k-1)+t})$ -stopping times $\rho^N(k) = \tau^N(k) - \tau^N(k-1)$ have the following uniform integrability property: the family $N\rho^N(k)$, $1 \leq k \leq N$, $N \in \mathbb{N}$, is uniformly integrable.*

Proof. Choose $\rho^N(k)$ according to the construction method outlined in the Appendix. More precisely, suppose that $\rho^N(k) = \Delta(M_{\tau^N(k-1)})$ (see last line of the Appendix for the definition). We now show that the family $N\rho^N(k)$, $1 \leq k \leq N$, $N \in \mathbb{N}$, is uniformly integrable.

Below, for random variables ξ and ζ , we write $\xi \stackrel{d}{=} \zeta$ (resp. $\xi \stackrel{d}{\leq} \zeta$) to indicate that ξ and ζ have the same distribution (resp. ζ stochastically dominates ξ). Let $U < \infty$ be an upper bound for $|\eta|$. Then Lemma A.3 yields

$$\rho^N(1) = \Delta(M_0) \stackrel{d}{=} \int_0^1 \frac{a_N^2(m) b_x^2(s, \tilde{W}_s)}{\eta^2(a_N(m) b(s, \tilde{W}_s) + m)} ds \geq \frac{a_N^2(m)}{U^2} \int_0^1 b_x^2(s, \tilde{W}_s) ds.$$

Therefore, the random variable $X = \int_0^1 b_x^2(s, W_s) ds$ is integrable. Now let $L > 0$ be a lower bound for $|\eta|$. Then, with \tilde{M}_0 having the distribution of $M_{\tau^N(k)}$, we get

$$\rho^N(k+1) = \Delta(M_{\tau^N(k)}) \stackrel{d}{=} \int_0^1 \frac{a_N^2(\tilde{M}_0) b_x^2(s, \tilde{W}_s)}{\eta^2(a_N(\tilde{M}_0) b(s, \tilde{W}_s) + \tilde{M}_0)} ds \stackrel{d}{\leq} \frac{A^2}{L^2 N} X,$$

where we also use that, by Lemma 3.2, we have $a_N(y) \leq \frac{A}{\sqrt{N}}$ for all $y \in I$. Thus,

$$P(N\rho^N(k+1) \geq x) \leq P\left(\frac{A^2}{L^2} X \geq x\right), \quad x \geq 0.$$

In other words, the integrable random variable $\frac{A^2}{L^2} X$ stochastically dominates every $N\rho^N(k)$, and hence we get the result. \square

We next aim at showing that for $s \in [0, 1]$ the stopping times $\tau^N(\lfloor Ns \rfloor)$ converge to s in probability. To this end we use the following version of the weak law of large numbers.

Theorem 3.4 (Weak LLN for uncorrelated arrays). *Let $(Z_k^n)_{n \in \mathbb{N}, 1 \leq k \leq n}$ be a triangular array of nonnegative and uniformly integrable random variables. Suppose that, for all $n \in \mathbb{N}$, the collection Z_k^n , $1 \leq k \leq n$, is pairwise uncorrelated. Then $\frac{1}{n} \sum_{k=1}^n (Z_k^n - EZ_k^n)$ converges to zero in probability.*

Proof. Let us set $\bar{Z}_k^n = Z_k^n 1_{\{Z_k^n \leq n\}}$ and define the sums $S_n = \sum_{k=1}^n Z_k^n$ and $\bar{S}_n = \sum_{k=1}^n \bar{Z}_k^n$. Since the family $(Z_k^n)_{n,k}$ is uniformly integrable, we have

$$C := \sup_{n \in \mathbb{N}, 1 \leq k \leq n} EZ_k^n < \infty, \quad (32)$$

$$D(n) := \sup_{1 \leq k \leq n} EZ_k^n 1_{\{Z_k^n > n\}} \rightarrow 0, \quad n \rightarrow \infty. \quad (33)$$

We need to prove that $\frac{S_n - ES_n}{n}$ converges to zero in probability. It follows from (33) and the estimates

$$\begin{aligned} P(S_n \neq \bar{S}_n) &\leq \sum_{k=1}^n P(Z_k^n > n) \leq n \sup_{1 \leq k \leq n} P(Z_k^n > n) \leq D(n), \\ 0 &\leq \frac{ES_n - E\bar{S}_n}{n} = \frac{1}{n} \sum_{k=1}^n EZ_k^n 1_{\{Z_k^n > n\}} \leq D(n) \end{aligned}$$

that it is enough to prove that $\frac{\bar{S}_n - E\bar{S}_n}{n}$ converges to zero in probability. We will now prove that the latter sequence converges to zero in L^2 .

We have

$$\begin{aligned} E \left(\frac{\bar{S}_n - E\bar{S}_n}{n} \right)^2 &= \frac{1}{n^2} \text{Var } \bar{S}_n \\ &\leq \frac{1}{n^2} \left(\sum_{k=1}^n E(\bar{Z}_k^n)^2 + 2 \sum_{1 \leq k < l \leq n} \text{Cov}(\bar{Z}_k^n, \bar{Z}_l^n) \right). \end{aligned} \quad (34)$$

Due to the uniform integrability of $(Z_k^n)_{n,k}$,

$$G(y) := \sup_{n \in \mathbb{N}, 1 \leq k \leq n} EZ_k^n 1_{\{Z_k^n > y\}} \rightarrow 0, \quad y \rightarrow \infty.$$

Since $E(\bar{Z}_k^n)^2 = \int_0^\infty 2yP(\bar{Z}_k^n > y) dy \leq \int_0^\infty 2yP(Z_k^n > y) dy \leq \int_0^\infty 2G(y) dy$, we get

$$\frac{1}{n^2} \sum_{k=1}^n E(\bar{Z}_k^n)^2 \leq \frac{1}{n} \int_0^\infty 2G(y) dy \rightarrow 0, \quad n \rightarrow \infty. \quad (35)$$

Using that the random variables Z_k^n , $1 \leq k \leq n$, are pairwise uncorrelated, we get, for $k \neq l$,

$$\begin{aligned} \text{Cov}(\bar{Z}_k^n, \bar{Z}_l^n) &= E\bar{Z}_k^n \bar{Z}_l^n - E\bar{Z}_k^n E\bar{Z}_l^n \leq EZ_k^n Z_l^n - E\bar{Z}_k^n E\bar{Z}_l^n \\ &= EZ_k^n EZ_l^n - (EZ_k^n - EZ_k^n 1_{\{Z_k^n > n\}})(EZ_l^n - EZ_l^n 1_{\{Z_l^n > n\}}) \\ &\leq C(EZ_k^n 1_{\{Z_k^n > n\}} + EZ_l^n 1_{\{Z_l^n > n\}}) \leq 2CD(n), \end{aligned}$$

where C is the constant from (32). Hence, $\limsup_{n \rightarrow \infty} \sup_{1 \leq k < l \leq n} \text{Cov}(\bar{Z}_k^n, \bar{Z}_l^n) = 0$. Together with (35) and the fact that the right-hand side of (34) is nonnegative, this implies that the right-hand side of (34) converges to zero. The proof is completed. \square

Observe that (15) implies that the sequence $(\rho^N(k))$ is pairwise uncorrelated and $E[\rho^N(k+1)|\mathcal{F}_{\tau^N(k)}] = \frac{1}{N}$.

Lemma 3.5. *Suppose that the family $(N\rho^N(k))_{N \in \mathbb{N}, 1 \leq k \leq N}$ is uniformly integrable. Then for all $s \in [0, 1]$ we have $\tau^N(\lfloor Ns \rfloor) \rightarrow s$ in probability.*

Proof. Let $s \in [0, 1]$. Set $Z_k^N = N\rho^N(k)$ if $k \leq \lfloor Ns \rfloor$ and $Z_k^N = 0$ else. Notice that the family $(Z_k^N)_{1 \leq k \leq N}$ satisfies the assumptions of Theorem 3.4, and hence $\frac{1}{N} \sum_{k=1}^N (Z_k^N - E(Z_k^N))$ converges to zero in probability. Notice that

$$\frac{1}{N} \sum_{k=1}^N Z_k^N = \sum_{k=1}^{\lfloor Ns \rfloor} \rho^N(k) = \tau^N(\lfloor Ns \rfloor),$$

and $\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N E(Z_k^N) = \lim_{N \rightarrow \infty} \frac{1}{N} \lfloor Ns \rfloor = s$. Consequently, $\tau(\lfloor Ns \rfloor)$ converges to s in probability, as $N \rightarrow \infty$. \square

The final arguments for proving Theorem 3.1 are now standard (cf. Section 8.6 in [4]). We denote by $\|\cdot\|_{C[0,1]}$ the sup norm in $C[0, 1]$.

Proof of Theorem 3.1. We can assume that $Y_k^N = M_{\tau^N(k)}$ and that the family $N\rho^N(k)$, $N \in \mathbb{N}$, $1 \leq k \leq N$, is uniformly integrable (see Lemma 3.3). Recall that $Y_{Nt}^N = Y_{\lfloor Nt \rfloor}^N + (Nt - \lfloor Nt \rfloor)(Y_{\lfloor Nt \rfloor + 1}^N - Y_{\lfloor Nt \rfloor}^N)$ for $t \in [0, 1]$.

First we show that $\|Y_{Nt}^N - M\|_{C[0,1]} \rightarrow 0$ in probability. To this end let $\varepsilon > 0$. For $\delta > 0$ let

$$A(\delta) = \{|M_t - M_s| < \frac{\varepsilon}{2} \text{ for all } t, s \in [0, 1] \text{ such that } |t - s| \leq 2\delta\}.$$

We choose δ such that $\frac{1}{\delta} \in \mathbb{N}$ and $P(A(\delta)) > 1 - \frac{\varepsilon}{2}$. Next we define

$$C(N, \delta) = \{|\tau^N(\lfloor Nk\delta \rfloor) - k\delta| \leq \delta \text{ for } k = 1, \dots, \frac{1}{\delta}\}.$$

By Lemma 3.5 there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ we have $P(C(N, \delta)) > 1 - \frac{\varepsilon}{2}$.

Notice that on the event $C(N, \delta)$ we have $|\tau^N(\lfloor Ns \rfloor) - s| \leq 2\delta$ for all $s \in [0, 1]$. In the following suppose that $A(\delta) \cap C(N, \delta)$ occurs. Then for $s = \frac{m}{N}$ we have $|\tau^N(m) - \frac{m}{N}| \leq 2\delta$ and hence

$$|Y_{Ns}^N - M_s| = |Y_m^N - M_{\frac{m}{N}}| < \frac{\varepsilon}{2}.$$

Let now $s \in (\frac{m}{N}, \frac{m+1}{N})$. Set $\theta = s - \frac{m}{N}$ and notice that for all $N \geq \frac{1}{2\delta}$

$$\begin{aligned} |Y_{Ns}^N - M_s| &\leq \theta |Y_m^N - M_{\frac{m}{N}}| + (1 - \theta) |Y_{m+1}^N - M_{\frac{m+1}{N}}| \\ &\quad + \theta |M_{\frac{m}{N}} - M_s| + (1 - \theta) |M_{\frac{m+1}{N}} - M_s| < \varepsilon. \end{aligned}$$

Consequently, for all $N \geq N_0 \vee \frac{1}{2\delta}$ we have $P(\|Y_N^N - M\|_{C[0,1]} > \varepsilon) < \varepsilon$. Since ε is arbitrary, we obtain that $\|Y_N^N - M\|_{C[0,1]} \rightarrow 0$ in probability.

To complete the proof, let $\psi: C[0, 1] \rightarrow \mathbb{R}$ be a bounded function that is continuous with respect to the sup norm. It is straightforward to show that $\lim_{N \rightarrow \infty} E\psi(Y_N^N) = E\psi(M)$, and hence the theorem is proved. \square

With a localization argument we can relax the assumption on η :

(C2) $|\eta|$ and $\frac{1}{|\eta|}$ are locally bounded on I .

Theorem 3.6. *Suppose (C2) and that μ has a compact support. Then the processes $(Y_{Nt}^N)_{t \in [0,1]}$ converge to $(M_t)_{t \in [0,1]}$ in distribution, as $N \rightarrow \infty$.*

Proof. The idea is first to redefine the times $\rho^N(k)$ to make sure that the family $N\rho^N(k)$, $N \in \mathbb{N}$, $1 \leq k \leq N$, is uniformly integrable. To this end choose a sequence of bounded intervals $[l_n, r_n] \subset I$ such that $l_n \downarrow l$ and $r_n \uparrow r$. Let $H(l_n, r_n)$ denote the first exit time of M from (l_n, r_n) . For a fixed n let $\hat{\rho}^N(k+1) = \frac{1}{N}$ if $\tau^N(k) > H(l_n, r_n)$, and let $\hat{\rho}^N(k+1) = \rho^N(k+1)$ otherwise. We set $\hat{\tau}^N(k) = \sum_{j=1}^k \hat{\rho}^N(j)$. Notice that $(\hat{\tau}^N(k))$ and $(\hat{\rho}^N(k))$ depend on the localizing parameter n .

Next observe that $\lim_{N \rightarrow \infty} \sup_{y \in (l_n, r_n)} a_N(y) = 0$. Indeed, by (20), we have

$$\int q(y, y + a_N(y)x) \mu(dx) \leq \frac{1}{N}. \quad (36)$$

Fatou's lemma yields $\int q(y, y + \liminf_{N \rightarrow \infty} a_N(y)x) \mu(dx) = 0$. Since $\mu \neq \delta_0$ and, clearly, the sequence $\{a_N(y)\}_{N \in \mathbb{N}}$ is decreasing, we obtain $\lim_{N \rightarrow \infty} a_N(y) = 0$. Performing two changes of variables in (36) leads to

$$Na_N^2(y) \int \int_0^x \int_0^u \frac{2}{\eta^2(y + a_N(y)r)} dr du \mu(dx) \leq 1.$$

Using Fatou's lemma again and taking into account the already established relation $\lim_{N \rightarrow \infty} a_N(y) = 0$, we get

$$\limsup_{N \rightarrow \infty} N a_N^2(y) \leq \frac{1}{\int \int_0^x \int_0^u \liminf_{N \rightarrow \infty} \frac{2}{\eta^2(y + a_N(y)r)} dr du \mu(dx)} \leq \frac{\limsup_{z \rightarrow y} \eta^2(z)}{\int x^2 \mu(dx)}.$$

From (C2) we deduce that $\lim_{N \rightarrow \infty} \sup_{y \in (l_n, r_n)} a_N(y) = 0$.

Since μ has a compact support one can show by an adaptation of Lemma 3.3 that, for each fixed n , the family $(N \hat{\rho}^N(k))_{N \in \mathbb{N}, 1 \leq k \leq N}$ is uniformly integrable. Lemma 3.5 implies that $\hat{\tau}^N(\lfloor Ns \rfloor) \rightarrow s$ in probability for all $s \in [0, 1]$. Let $\hat{Y}_k^N = M_{\hat{\tau}^N(k)}$ and $Y_k^N = M_{\tau^N(k)}$. As in the proof of Theorem 3.1 one can show that $\|\hat{Y}_N^N - M\|_{C[0,1]} \rightarrow 0$ in probability.

Let us consider several cases. First let $q(l+) = q(r-) = \infty$, that is, both endpoints l and r are inaccessible. Fix $\varepsilon > 0$ and choose n such that $P(H(l_n, r_n) \geq 2) > 1 - \frac{\varepsilon}{2}$. Then we choose N_0 such that, for any $N \geq N_0$, we have $P(\hat{\tau}^N(N) \leq 2) > 1 - \frac{\varepsilon}{2}$. On the event $\{H(l_n, r_n) \geq 2, \hat{\tau}^N(N) \leq 2\}$ of probability at least $1 - \varepsilon$ we have $Y_k^N = \hat{Y}_k^N$ for all $0 \leq k \leq N$. Thus, $\|Y_N^N - M\|_{C[0,1]} \rightarrow 0$ in probability.

Let now $q(l+) < \infty$ and $q(r-) = \infty$, i.e. l is accessible, r is inaccessible. In this case, $l > -\infty$ and $H(l_n, r_n) \rightarrow H(l)$ a.s. as $n \rightarrow \infty$, where $H(l)$ denotes the hitting time of l by the process M . Fix $\varepsilon > 0$ and choose n such that $P(A_n) < \frac{\varepsilon}{3}$ and $P(B_n) < \frac{\varepsilon}{3}$ with

$$\begin{aligned} A_n &= \{H(l) \leq 3 \text{ and } \exists s \in [H(l_n, r_n), H(l)] \text{ such that } M_s > l + \varepsilon\}, \\ B_n &= \{H(l) > 3 \text{ and } H(l_n, r_n) < 2\}. \end{aligned}$$

Then choose N_0 such that, for any $N \geq N_0$, we have $P(\hat{\tau}^N(N) \leq 2) > 1 - \frac{\varepsilon}{3}$. Given an event A , by A^c we denote the complement of A . For $N \geq N_0$, on the event

$$\{\hat{\tau}^N(N) \leq 2\} \cap (A_n \cup B_n)^c$$

of probability at least $1 - \varepsilon$ we have either

$$H(l) > 3, H(l_n, r_n) \geq 2, \text{ hence } Y_k^N = \hat{Y}_k^N \text{ for all } 0 \leq k \leq N,$$

or

$$\begin{aligned} &H(l) \leq 3, M_s \in [l, l + \varepsilon] \text{ for } s \in [H(l_n, r_n), H(l)], \\ &\text{hence } |Y_{Ns}^N - M_s| \leq \varepsilon \text{ whenever } Y_{Ns}^N \neq \hat{Y}_{Ns}^N, s \in [0, 1]. \end{aligned}$$

Thus, $\|Y_N^N - M\|_{C[0,1]} \rightarrow 0$ in probability.

The remaining cases are considered in a similar way. □

We can also combine the boundedness assumptions on η and on the support of μ in other ways:

Theorem 3.7. *Assume that, for any $y \in I$, $|\eta|$ and $\frac{1}{|\eta|}$ are bounded on (l, y) (the bounds may depend on y) and that $\sup \text{supp } \mu < \infty$. Then the processes $(Y_{Nt}^N)_{t \in [0,1]}$ converge to $(M_t)_{t \in [0,1]}$ in distribution, as $N \rightarrow \infty$.*

The proof is similar to that of Theorem 3.6. Clearly, Theorem 3.7 has its analogue “at r ”.

Examples

We close the section by illustrating our results with several examples.

Example 3.8 (Brownian motion). Let M be a Brownian motion starting from some $m \in \mathbb{R}$, i.e. we have $l = -\infty$, $r = \infty$ and $\eta \equiv 1$. Then $q(y, x) = (x - y)^2$, for $y, x \in \mathbb{R}$, and

$$G_y(a) = a^2 \int x^2 \mu(dx), \quad y \in \mathbb{R}, \quad a \geq 0.$$

Therefore, condition (A1) of Section 2.1 is satisfied if and only if $\sigma^2 := \int x^2 \mu(dx) < \infty$. In this case, the scaled random walk (Y_k^N) is determined by the scale factor

$$a_N(y) = \frac{1}{\sqrt{N\sigma^2}},$$

which does not depend on the state y . Thus, since (C1) is satisfied, Theorem 3.1 yields weak convergence of (Y_{Nt}^N) to (M_t) under the assumptions that $\mu \neq \delta_0$ is centered and $\int x^2 \mu(dx) < \infty$. This is exactly the Donsker–Prokhorov invariance principle.

Example 3.9 (Diffusion between two media). Let $l = -\infty$, $r = \infty$ and, with some $A \in \mathbb{R} \setminus \{0\}$,

$$\eta(x) = 1_{(0, \infty)}(x) + A 1_{(-\infty, 0]}(x), \quad x \in \mathbb{R}.$$

Notice that we have

$$\begin{aligned} \text{for } y \geq 0 : \quad q(y, x) &= \begin{cases} (x - y)^2, & x \geq 0, \\ y^2 - 2xy + \frac{1}{A^2}x^2, & x < 0, \end{cases} \\ \text{for } y \leq 0 : \quad q(y, x) &= \begin{cases} \frac{1}{A^2}(x - y)^2, & x < 0, \\ \frac{1}{A^2}y^2 - \frac{2}{A^2}xy + x^2, & x \geq 0. \end{cases} \end{aligned}$$

Since, for appropriate $0 < c_1 < c_2 < \infty$, we have $c_1(x - y)^2 \leq q(y, x) \leq c_2(x - y)^2$, condition (A1) is satisfied if and only if μ has a finite second moment. Again, (C1) is satisfied, hence the processes (Y_{Nt}^N) converge in distribution to (M_t) for any such μ .

Example 3.10 (Geometric Brownian motion). Let $l = 0$, $r = \infty$ and $\eta(x) = x$ on I . For $y > 0$, we have

$$q(y, x) = \begin{cases} 2\frac{x-y}{y} - 2\log \frac{x}{y}, & \text{if } x > 0, \\ \infty, & \text{if } x \leq 0. \end{cases}$$

Since, for fixed $y > 0$, $q(y, x)$ has linear growth as $x \rightarrow \infty$, condition (A2) of Section 2.2 is satisfied if and only if $\inf \text{supp } \mu > -\infty$. For all such measures μ , (19) is satisfied due to Proposition 2.13, and hence Theorem 2.7 applies; that is, for sufficiently large $N \in \mathbb{N}$, Problem (P) has a solution with scale factor a_N satisfying (17). Since (C2) holds true, by Theorem 3.6, the processes (Y_{Nt}^N) converge in distribution to (M_t) for any μ with a compact support.

A. Appendix

We use the setting and notations of Section 1. In particular, we consider a weak solution (M, W) of (6), where the initial condition M_0 has distribution γ , and we treat the embedding problem (9), where $a: I \rightarrow (0, \infty)$ is a given Borel function. Let us now briefly explain, following [1], a solution method of (9), which gives an embedding stopping time satisfying (12) provided (11) holds true.

Let \tilde{W} be an $(\tilde{\mathcal{F}}_t)$ -Brownian motion on some $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{P})$ and \tilde{M}_0 an $\tilde{\mathcal{F}}_0$ -measurable random variable with distribution γ . For $y \in I$, let F_y and F_μ be the distribution functions of $K(y, a(y), \cdot)$ and of μ , as well as F_y^{-1} and F_μ^{-1} their generalized inverse functions (that is, $F_y^{-1}(r) = \inf\{x \in \mathbb{R} : F_y(x) > r\}$, $r \in (0, 1)$, and the same formula holds for F_μ^{-1}). For $y \in I$, $t \in [0, 1]$ and $x \in \mathbb{R}$, we define

$$\begin{aligned} g(y, t, x) &= \tilde{E}[F_y^{-1} \circ \Phi(\tilde{W}_1) | \tilde{W}_t = x], \\ b(t, x) &= \tilde{E}[F_\mu^{-1} \circ \Phi(\tilde{W}_1) | \tilde{W}_t = x], \end{aligned}$$

where Φ denotes the standard normal distribution function, and notice that

$$g(y, t, x) = y + a(y)b(t, x). \quad (37)$$

Let us define the $(\tilde{\mathcal{F}}_t)$ -martingale $N_t = b(t, \tilde{W}_t)$, $t \in [0, 1]$, and the process $L_t = g(\tilde{M}_0, t, \tilde{W}_t) = \tilde{M}_0 + a(\tilde{M}_0)N_t$, $t \in [0, 1]$ (the latter process can fail to be a martingale because it can fail to be integrable). Observe that N_1 has the distribution μ , hence

$$\text{Law}(L_1 | \tilde{\mathcal{F}}_0) = K(\tilde{M}_0, a(\tilde{M}_0), \cdot). \quad (38)$$

Moreover, we have

$$\tilde{P}((L_t)_{t \in [0, 1]} \in A | \tilde{\mathcal{F}}_0) = G(\tilde{M}_0, A), \quad A \in \mathcal{B}(C[0, 1]), \quad (39)$$

where the kernel G is given by the formula

$$G(y, A) = \tilde{P}((g(y, t, \tilde{W}_t))_{t \in [0, 1]} \in A), \quad y \in I, \quad A \in \mathcal{B}(C[0, 1]). \quad (40)$$

One can also check that the function b is smooth on $[0, 1] \times \mathbb{R}$ and, for any $t \in [0, 1]$, the function $b(t, \cdot)$ is a strictly increasing bijective mapping $\mathbb{R} \rightarrow (\inf \text{supp } \mu, \sup \text{supp } \mu)$. Let g^{-1} denote the inverse of g in the last argument, which is well defined when the second argument $t \in [0, 1]$.

A straightforward generalization of Theorems 1 and 3 and Lemma 2 in [1] now yields the following statement.

Proposition A.1. *Assume that (11) holds true. Then the ODE*

$$\delta'(t) = \frac{a^2(M_0)b_x^2(t, g^{-1}(M_0, t, M_{\delta(t)}))}{\eta^2(M_{\delta(t)}), \quad t \in [0, 1], \quad \delta(0) = 0, \quad (41)$$

has a solution on $[0, 1)$ for P -almost all paths. Here, b_x denotes the partial derivative of b with respect to the second argument. We set

$$\delta(1) = \lim_{t \uparrow 1} \delta(t), \quad (42)$$

which is well defined P -a.s. because δ is nondecreasing. Moreover, $(\delta(t))_{t \in [0, 1]}$ is an (\mathcal{F}_t) -time change, the (\mathcal{F}_t) -stopping time $\delta(1)$ satisfies

$$E[\delta(1) | \mathcal{F}_0] = Q(M_0) \quad P\text{-a.s.}, \quad (43)$$

the process

$$Z_t = \frac{1}{a(M_0)}(M_{\delta(t)} - M_0), \quad t \in [0, 1], \quad (44)$$

is an $(\mathcal{F}_{\delta(t)})$ -martingale, and

$$\text{Law}(Z_t; t \in [0, 1] | \mathcal{F}_0) = \text{Law}(N_t; t \in [0, 1]) \quad P\text{-a.s.}, \quad (45)$$

where the left-hand side is the notation for the regular conditional distribution of the process $(Z_t)_{t \in [0, 1]}$ with respect to \mathcal{F}_0 , while the right-hand side is the notation for the unconditional distribution of the process $(N_t)_{t \in [0, 1]}$ (that is, the former, which is in general a kernel depending on ω , equals the latter for almost all paths).

Corollary A.2. Assume that (11) holds true. Then

$$\text{Law}(M_{\delta(t)}; t \in [0, 1] | \mathcal{F}_0) = G(M_0, \cdot) \quad P\text{-a.s.}, \quad (46)$$

where the kernel G is given by (40) (recall the relation between the processes (N_t) and (L_t) right after (37)). In particular, $\delta(1)$ is a solution of the embedding problem (9) satisfying (43) (see (38) and (39)).

The next lemma summarizes the properties we need in this paper.

Lemma A.3. Assume (11). Then the following holds true.

(i) The process

$$X_t = \frac{1}{a(M_0)}(M_{\delta(1) \wedge t} - M_0), \quad t \geq 0,$$

is a uniformly integrable (\mathcal{F}_t) -martingale.

(ii) The (\mathcal{F}_t) -stopping time $\delta(1)$ has the same distribution as the random variable

$$\xi = \int_0^1 \frac{a^2(\tilde{M}_0) b_x^2(s, \tilde{W}_s)}{\eta^2(\tilde{M}_0 + a(\tilde{M}_0) b(s, \tilde{W}_s))} ds. \quad (47)$$

(Of course one can drop the tildes in the latter formula.)

Proof. (i) First observe that the (\mathcal{F}_t) -time change $(\delta(t))_{t \in [0,1]}$ is P -a.s. strictly increasing on $[0, 1]$. Indeed, if it had an interval of constancy, then, by (44) and (45), the process $(N_t)_{t \in [0,1]}$ would have an interval of constancy, which is impossible because $N_t = b(t, \tilde{W}_t)$ and, for $t \in [0, 1)$, b is smooth in both arguments and $b(t, \cdot)$ is strictly increasing. Thus, the inverse δ^{-1} is well defined.

Now, for a fixed $t \geq 0$, define $\eta = \delta^{-1}(\delta(1) \wedge t)$. Since $\delta(1) \wedge t$ is an (\mathcal{F}_t) -stopping time, η is an $(\mathcal{F}_{\delta(t)})$ -stopping time. Clearly, $\eta \leq 1$. Doob's optional sampling theorem applied to the $(\mathcal{F}_{\delta(t)})$ -martingale $(Z_t)_{t \in [0,1]}$ (see (44)) and to the bounded $(\mathcal{F}_{\delta(t)})$ -stopping times η and 1 yields $E(Z_1 | \mathcal{F}_{\delta(\eta)}) = Z_\eta$, P -a.s., which is equivalent to

$$E(X_\infty | \mathcal{F}_{\delta(1) \wedge t}) = X_t \quad P\text{-a.s.}$$

A short calculation reveals that, since the process $(X_t)_{t \geq 0}$ is stopped at $\delta(1)$, we also have

$$E(X_\infty | \mathcal{F}_t) = X_t \quad P\text{-a.s.}$$

This concludes the proof of (i).

(ii) Formulas (41), (46), (40), (39) as well as

$$g^{-1}(\tilde{M}_0, t, L_t) = \tilde{W}_t \quad \text{and} \quad L_t = \tilde{M}_0 + a(\tilde{M}_0)b(t, \tilde{W}_t)$$

immediately imply

$$\text{Law}(\delta(1) | \mathcal{F}_0) = H(M_0, \cdot) \quad P\text{-a.s.}, \quad (48)$$

where the kernel H is given by the formula

$$H(y, \cdot) = \text{Law} \left(\int_0^1 \frac{a^2(y)b_x^2(s, \tilde{W}_s)}{\eta^2(y + a(y)b(s, \tilde{W}_s))} ds \right), \quad y \in I.$$

Since \tilde{M}_0 is $\tilde{\mathcal{F}}_0$ -measurable and the process (\tilde{W}_s) is independent of $\tilde{\mathcal{F}}_0$, then for the random variable ξ of (47) we get

$$\text{Law}(\xi | \tilde{\mathcal{F}}_0) = H(\tilde{M}_0, \cdot) \quad \tilde{P}\text{-a.s.} \quad (49)$$

The statement now follows from (48), (49) and the fact that M_0 and \tilde{M}_0 have the same distribution. \square

Sometimes we use the notation $\Delta = \delta(1)$ and also write $\Delta(M_0)$ instead of Δ whenever we want to stress the dependence on M_0 .

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